N20/5/MATHL/HP1/ENG/TZ0/XX/M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2020

Mathematics

Higher level

Paper 1

30 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. $\mathrm{E}(X) = (0 \times p) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{6}\right) + 3q \left(= \frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow \frac{1}{4} + \frac{1}{3} + 3q = \frac{19}{12}\right)$

$q = \frac{1}{3}$ ***A1***

$p + \frac{1}{4} + \frac{1}{6} + q = 1$ ***(M1)***

$\left(\Rightarrow p + q = \frac{7}{12}\right)$

$p = \frac{1}{4}$ ***A1***

[4 marks]

2. $(x = 0 \Rightarrow)\ y = 1$ ***(A1)***

appreciate the need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

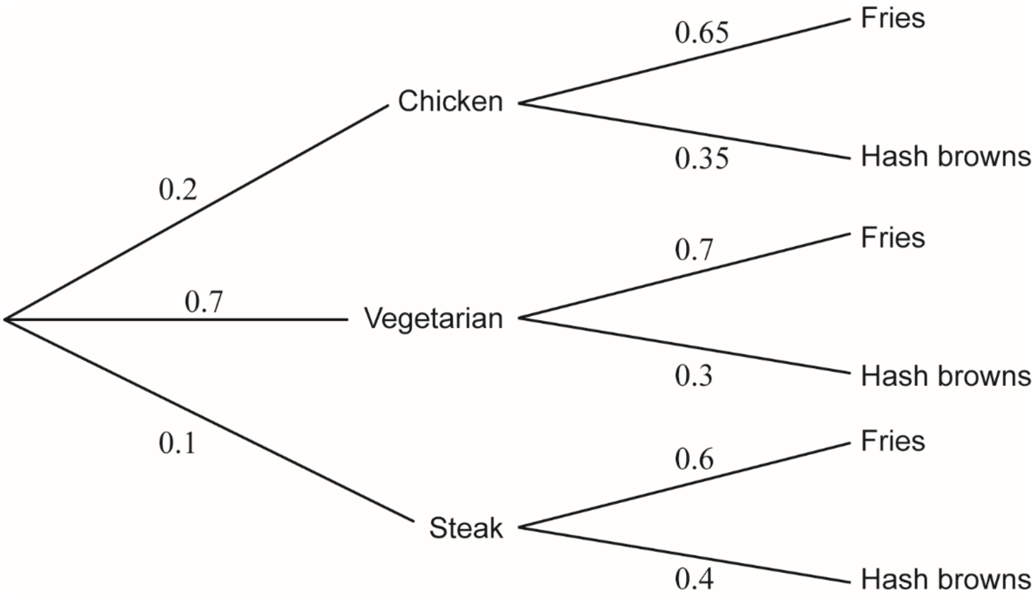
$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \right) 2\mathrm{e}^{2x} - 3$ ***A1***

$(x = 0 \Rightarrow) \frac{\mathrm{d}y}{\mathrm{d}x} = -1$ ***A1***

$\frac{y-1}{x-0} = -1 \ \ (y = 1 - x)$ ***A1***

[5 marks]

3. (a)



A1A1

> **Note:** Award ***A1*** for probabilities for type of omelette and ***A1*** for probabilities for fries / hash browns.

[2 marks]

(b) $(0.2\times0.65)+(0.7\times0.7)+(0.1\times0.6)$ ***(M1)***

$=0.68\left(=\frac{17}{25}\right)$ ***A1***

[2 marks]

(c) $\frac{\text{P}(\text{ordered fries and did not order chicken omelette})}{\text{P}(\text{ did not order chicken omelette})}$ ***(M1)***

$\frac{0.7\times0.7+0.1\times0.6}{0.7+0.1}\left(=\frac{0.49+0.06}{0.8}=\frac{0.55}{0.8}\right)$ ***(A1)***

$=\frac{55}{80}\left(=\frac{11}{16}\right)$ ***A1***

[3 marks]
Total [7 marks]

4. substituting $z = x + \mathrm{i}y$ **and** $z^{*} = x - \mathrm{i}y$ ***M1***

$$\frac{2(x+\mathrm{i}y)}{3-(x-\mathrm{i}y)} = \mathrm{i}$$

$$2x + 2\mathrm{i}y = -y + \mathrm{i}(3-x)$$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3 - x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = (3-x)^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1,\ y = 2\ \left(z = -1 + 2\mathrm{i}\right)$ ***A1A1***

[5 marks]

5. $u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2\left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2\left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2\left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2\left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

6. METHOD 1

$\sin\theta\cos\theta = \frac{c}{a}$ and $\sin\theta + \cos\theta = -\frac{b}{a}$ ***A1***

attempt to square $\sin\theta + \cos\theta$ ***M1***

$\left(\frac{b^2}{a^2} =\right)\left(\sin\theta + \cos\theta\right)^2 = 1 + 2\sin\theta\cos\theta$ ***A1***

$\frac{b^2}{a^2}\left(= 1 + 2\sin\theta\cos\theta\right) = 1 + \frac{2c}{a}$ ***A1***

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

METHOD 2

$a\sin^2\theta + b\sin\theta + c = 0$ and $a\cos^2\theta + b\cos\theta + c = 0$ ***A1***

adding the two equations ***M1***

$a + b\left(\sin\theta + \cos\theta\right) + 2c = 0$ ***A1***

$a + b \times -\frac{b}{a} + 2c = 0$ ***A1***

$a^2 - b^2 + 2ac = 0$

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

7. (a) (i) $\frac{z_1}{z_2} = \cos\left(\frac{11\pi}{12} - \frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12} - \frac{\pi}{6}\right)$ ***(M1)***

$= \cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}$ ***A1***

(ii) $\frac{z_2}{z_1} = \cos\frac{3\pi}{4} - \mathrm{i}\sin\frac{3\pi}{4}$ ***A1***

> **Note:** Allow equivalent forms in part (a), *e.g.* $\mathrm{cis}\left(-\frac{3\pi}{4}\right)$.
>
> **Note:** Ignore subsequent work once correct answer(s) are seen.

[3 marks]

(b) valid attempt to calculate area of their triangle ***(M1)***

(angle between OA and OB is $\frac{\pi}{2}$) $\Rightarrow$ area $\left(= \frac{1}{2} \times 1 \times 1\right) = \frac{1}{2}$ ***A1***

[2 marks]

8. (a) **METHOD 1**

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1 - \cos x} \equiv \frac{\sin^2 x}{\cos x(1 - \cos x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$= \frac{1 - \cos^2 x}{\cos x(1 - \cos x)}$$

$$= \frac{(1 + \cos x)(1 - \cos x)}{\cos x(1 - \cos x)}$$ ***A1***

$$= \frac{(1 + \cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

METHOD 2

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos x)(1+\cos x)} \equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos^2 x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x \sin^2 x}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

(b) **METHOD 1**

consider $1+\dfrac{1}{\cos x}=k$, leading to $\cos x=\dfrac{1}{k-1}$ ***(M1)***

consider graph of $y=\dfrac{1}{x-1}$ or range of solutions for $y=\cos x$ ***(M1)***

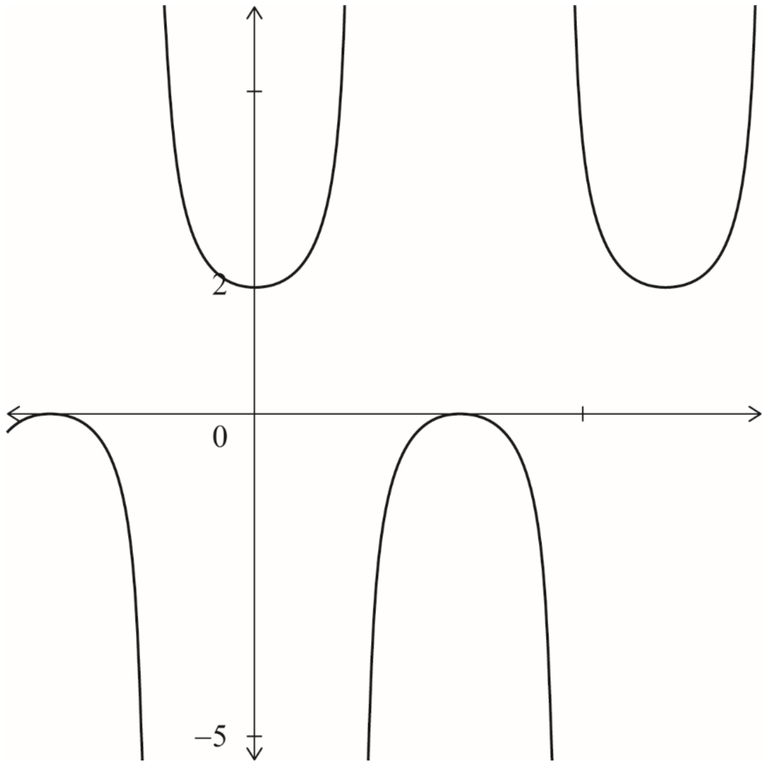
(no solutions if $y<-1$ or $y>1 \Rightarrow$) $0<k<2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k>0$ and $k<2$'.

[4 marks]

METHOD 2

consider graph of $y = 1 + \sec x$ ***M1***



A1

no real solutions if $0 < k < 2$ ***A1A1***

> **Note:** Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]

METHOD 3

consider $-1 \leq \cos x \leq 1$, ***(M1)***

$\frac{1}{\cos x} \leq -1$ or $\frac{1}{\cos x} \geq 1$ ***(M1)***

$1+\frac{1}{\cos x} \leq 0$ or $1+\frac{1}{\cos x} \geq 2$

no solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]
Total [7 marks]

9. $x = \tan u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \sec^2 u$ OR $u = \arctan x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2}$ ***A1***

attempt to write the integral in terms of u ***M1***

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \ \mathrm{d}u}{\left(1+\tan^2 u\right)^3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u \ \mathrm{d}u}{\left(\sec^2 u\right)^3}$$ ***(A1)***

$$= \int_0^{\frac{\pi}{4}} \sin^2 u \cos^2 u \ \mathrm{d}u$$ ***A1***

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2u \ \mathrm{d}u$$ ***M1***

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \left(1-\cos 4u\right) \ \mathrm{d}u$$ ***M1***

$$= \frac{1}{8}\left[u - \frac{\sin 4u}{4}\right]_0^{\frac{\pi}{4}}$$ ***A1***

$$= \frac{1}{8}\left[\frac{\pi}{4} - \frac{\sin \pi}{4} - 0 - 0\right]$$

$$= \frac{\pi}{32}$$ ***A1***

Total [8 marks]

Section B

10. (a) (i) $f'(x) = 3ax^2 + 2bx + c$ ***A1***

(ii) since f^{-1} does not exist, there must be two turning points ***R1***

($\Rightarrow f'(x) = 0$ has more than one solution)

using the discriminant $\Delta > 0$ ***M1***

$4b^2 - 12ac > 0$ ***A1***

$b^2 - 3ac > 0$ ***AG***

[4 marks]

(b) (i) **METHOD 1**

$b^2 - 3ac = (-3)^2 - 3 \times \frac{1}{2} \times 6$ ***M1***

$= 9 - 9$

$= 0$ ***A1***

hence g^{-1} exists ***AG***

METHOD 2

$g'(x) = \frac{3}{2}x^2 - 6x + 6$ ***M1***

$\Delta = (-6)^2 - 4 \times \frac{3}{2} \times 6$

$\Delta = 36 - 36 = 0 \Rightarrow$ there is (only) one point with gradient of 0
and this must be a point of inflexion (since $g(x)$ is a cubic.) ***R1***

hence g^{-1} exists ***AG***

(ii) $p = \frac{1}{2}$ ***A1***

$(x-2)^3 = x^3 - 6x^2 + 12x - 8$ ***(M1)***

$\frac{1}{2}(x^3 - 6x^2 + 12x - 8) = \frac{1}{2}x^3 - 3x^2 + 6x - 4$

$g(x) = \frac{1}{2}(x-2)^3 - 4 \Rightarrow q = -4$ ***A1***

(iii) $x = \frac{1}{2}(y-2)^3 - 4$ ***(M1)***

Note: Interchanging x and y can be done at any stage.

$2(x+4) = (y-2)^3$ ***(M1)***

$\sqrt[3]{2(x+4)} = y - 2$

$y = \sqrt[3]{2(x+4)} + 2$

$g^{-1}(x) = \sqrt[3]{2(x+4)} + 2$ ***A1***

Note: $g^{-1}(x) = \ldots$ must be seen for the final ***A*** mark.

[8 marks]

(c) translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ***A1***

Note: This can be seen anywhere.

EITHER

a stretch scale factor $\frac{1}{2}$ parallel to the y-axis then a translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ***A2***

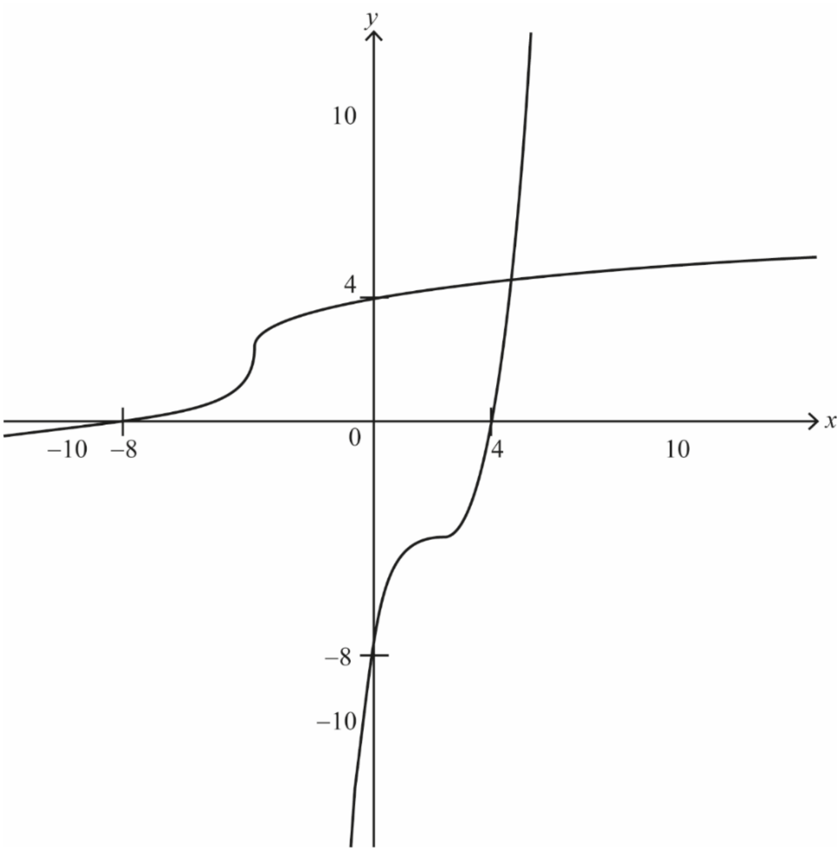
OR

a translation through $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ then a stretch scale factor $\frac{1}{2}$ parallel to the y-axis ***A2***

Note: Accept 'shift' for translation, but do not accept 'move'. Accept 'scaling' for 'stretch'.

[3 marks]

(d)



A1A1A1
M1A1

Note: Award ***A1*** for correct 'shape' of g (allow non-stationary point of inflexion)
Award ***A1*** for each correct intercept of g
Award ***M1*** for attempt to reflect their graph in $y = x$, ***A1*** for completely correct g^{-1} including intercepts

[5 marks]
Total [20 marks]

11. (a) attempt at implicit differentiation ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos(xy)\left[x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right]$$ ***A1M1A1***

> **Note:** Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy)+y\cos(xy)$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}-x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy)=y\cos(xy)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y-x\cos(xy)\right)=y\cos(xy)$$ ***M1***

> **Note:** Award ***M1*** for collecting derivatives and factorising.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y\cos(xy)}{2y-x\cos(xy)}$$ ***AG***

[5 marks]

(b) setting $\frac{dy}{dx} = 0$

$y\cos(xy) = 0$ ***(M1)***

$(y \neq 0) \Rightarrow \cos(xy) = 0$ ***A1***

$\Rightarrow \sin(xy)\left(= \pm\sqrt{1-\cos^2(xy)} = \pm\sqrt{1-0}\right) = \pm 1$ ***OR*** $xy = (2n+1)\frac{\pi}{2}\ (n \in \mathbb{Z})$

OR $xy = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ ***A1***

Note: If they offer values for xy, award ***A1*** for at least two correct values in two different 'quadrants' and no incorrect values.

$y^2\left(= \sin(xy)\right) > 0$ ***R1***

$\Rightarrow y^2 = 1$ ***A1***

$\Rightarrow y = \pm 1$ ***AG***

[5 marks]

(c) $y = \pm 1 \Rightarrow 1 = \sin(\pm x) \Rightarrow \sin x = \pm 1$ OR $y = \pm 1 \Rightarrow 0 = \cos(\pm x) \Rightarrow \cos x = 0$ ***(M1)***

$(\sin x = 1 \Rightarrow)\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$ ***A1A1***

$(\sin x = -1 \Rightarrow)\left(\frac{3\pi}{2}, -1\right), \left(\frac{7\pi}{2}, -1\right)$ ***A1A1***

Note: Allow 'coordinates' expressed as $x = \frac{\pi}{2}, y = 1$ for example.

Note: Each of the ***A*** marks may be awarded independently and are not dependent on ***(M1)*** being awarded.

Note: Mark only the candidate's first two attempts for each case of $\sin x$.

[5 marks]

Total [15 marks]

12. (a) $x = k$ ***A1***

[1 mark]

(b) $y = k$ ***A1***

[1 mark]

(c) **METHOD 1**

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x(k^2-5)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

[4 marks]

METHOD 2

$$f\left(x\right)=\frac{kx-5}{x-k}$$

$$x=\frac{ky-5}{y-k}$$ ***M1***

Note: Interchanging x and y can be done at any stage.

$$x\left(y-k\right)=ky-5$$ ***A1***

$$xy-xk=ky-5$$

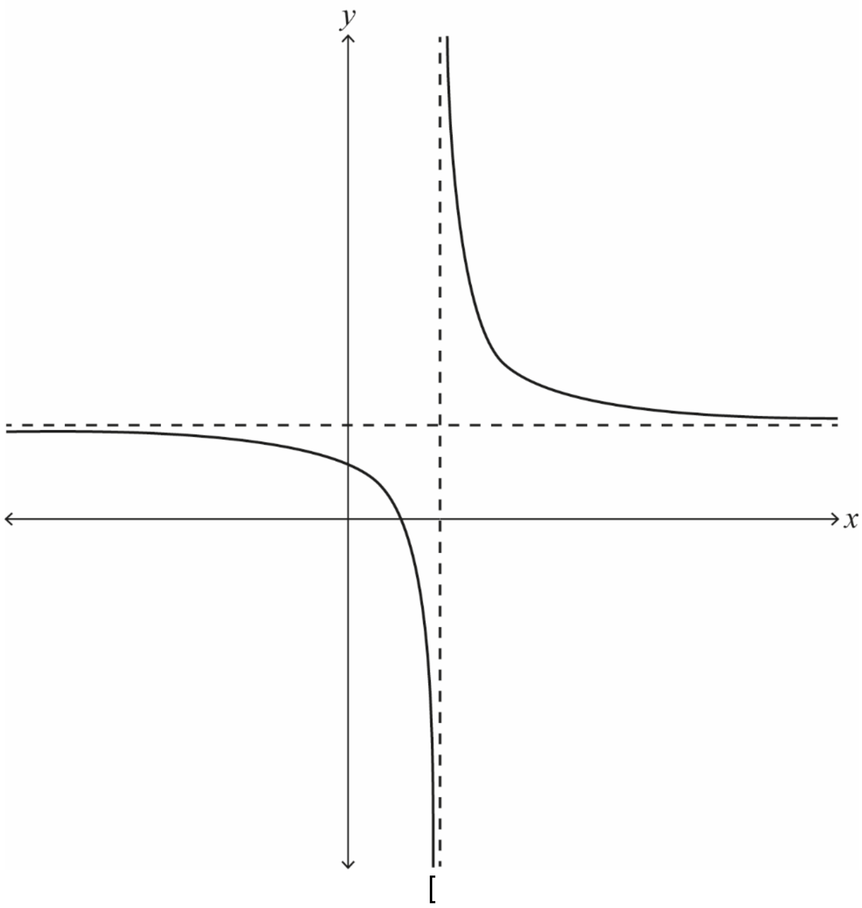
$$xy-ky=xk-5$$

$$y\left(x-k\right)=kx-5$$ ***A1***

$y=f^{-1}(x)=\frac{kx-5}{x-k}$ (hence f is self-inverse.) ***R1***

[4 marks]

(d)



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$ ***A1***

[3 marks]

(e) **METHOD 1**

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p+\frac{q}{x-3}$ ***M1***

$\frac{3x-5}{x-3} = 3+\frac{4}{x-3}$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24x-56}{(x-3)^2}$ ***A1***

THEN

$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24}{x-3}+\frac{16}{(x-3)^2}$ ***A1***

volume $= \pi\int_5^7 \left(9+\frac{24}{x-3}+\frac{16}{(x-3)^2}\right) \mathrm{d}x$

$= \pi\left[9x+24\ln(x-3)-\frac{16}{x-3}\right]_5^7$ ***A1***

$= \pi\left[(63+24\ln 4-4)-(45+24\ln 2-8)\right]$

$= \pi(22+24\ln 2)$ ***A1***

[6 marks]

METHOD 2

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ *(M1)*

substituting $u = x-3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ **A1**

$$3x-5 = 3(u+3)-5 = 3u+4$$

$$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$$ **M1**

$$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\mathrm{d}u$$ **A1**

$$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$$ **A1**

Note: Ignore absence of or incorrect limits seen up to this point.

$$= \pi\left(22 + 24\ln 2\right)$$ **A1**

[6 marks]
Total [15 marks]